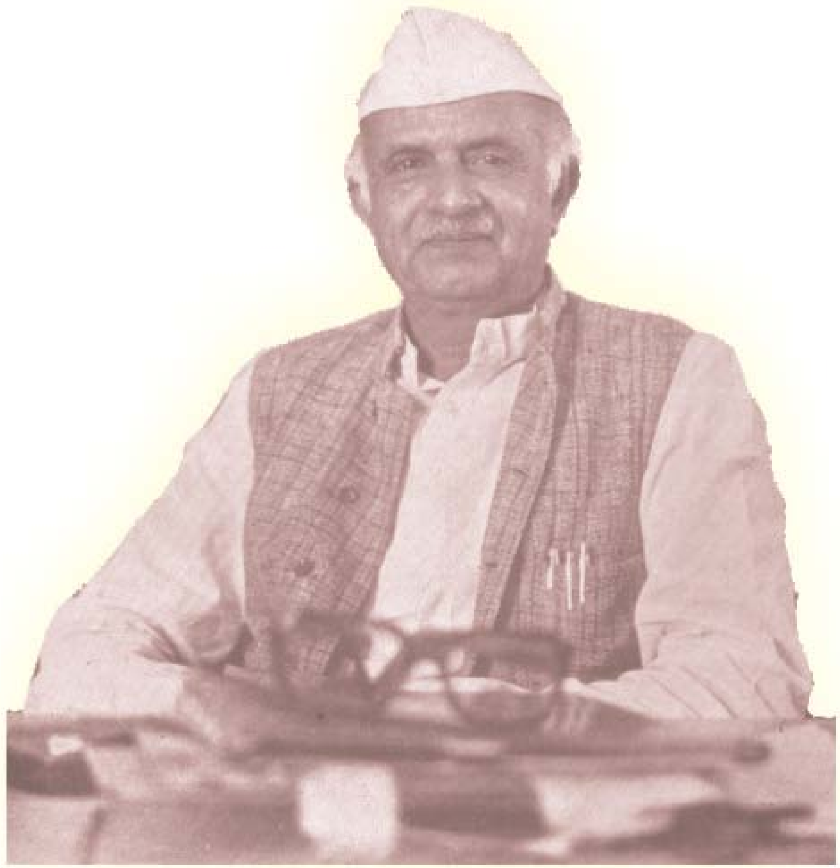
*Gravitational Waves- a new Window to Cosmos**

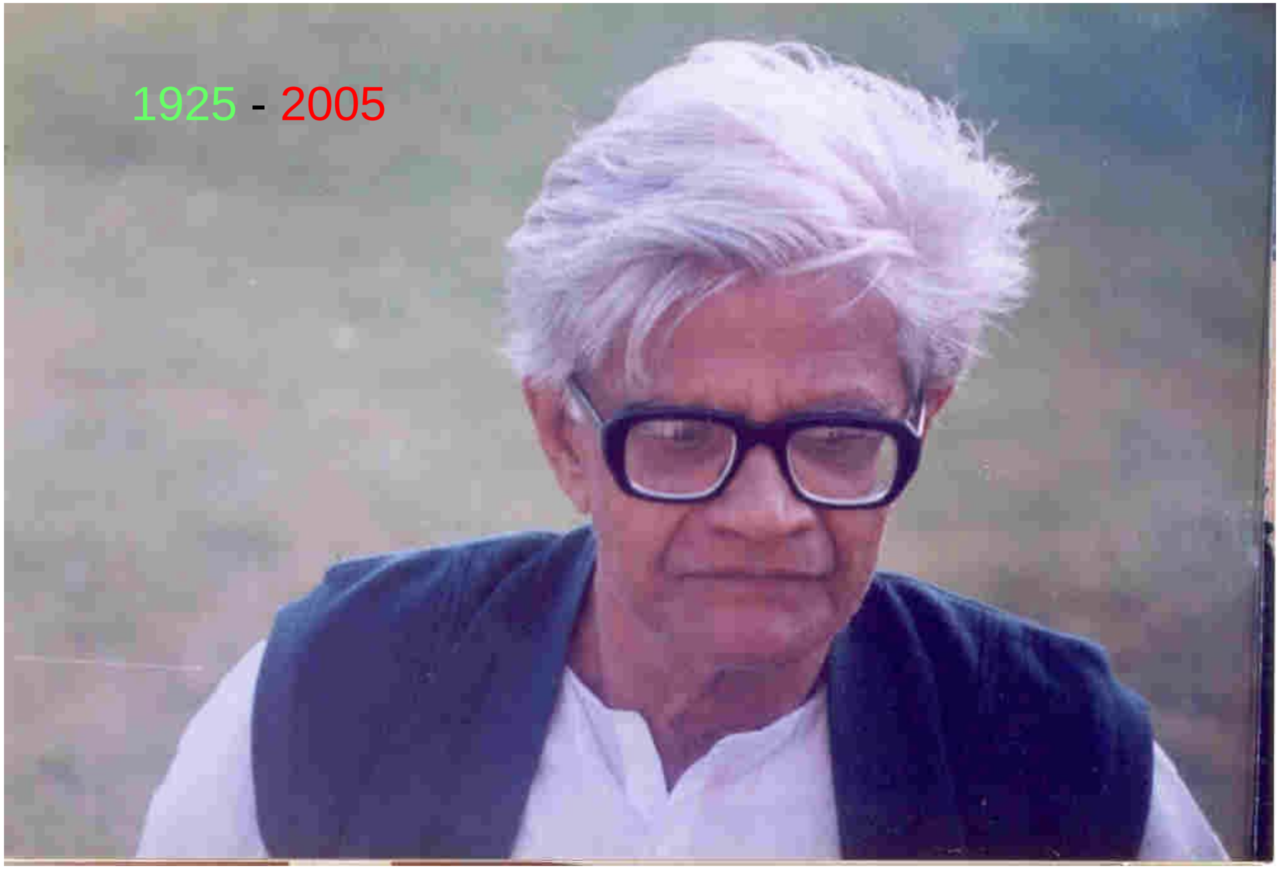
A.R.Prasanna,
Physical Research Laboratory,
Ahmedabad

- **IAGRG Vaidya-Raichaudhuri award lecture**
- **delivered at Lucknow University, Physics Department- 22.September,2016**

1918 - 2010



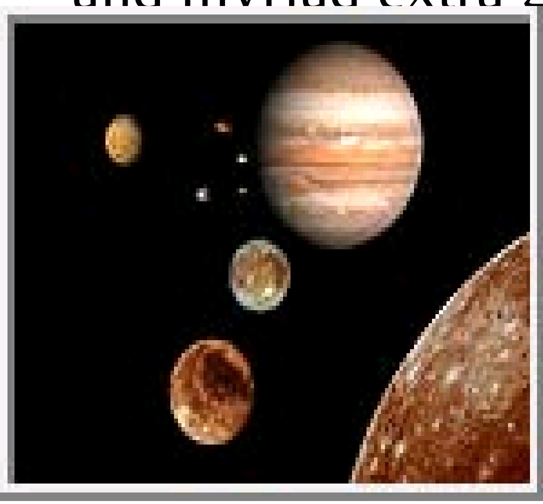
$$ds^2 = \frac{\dot{m}^2}{f^2}(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

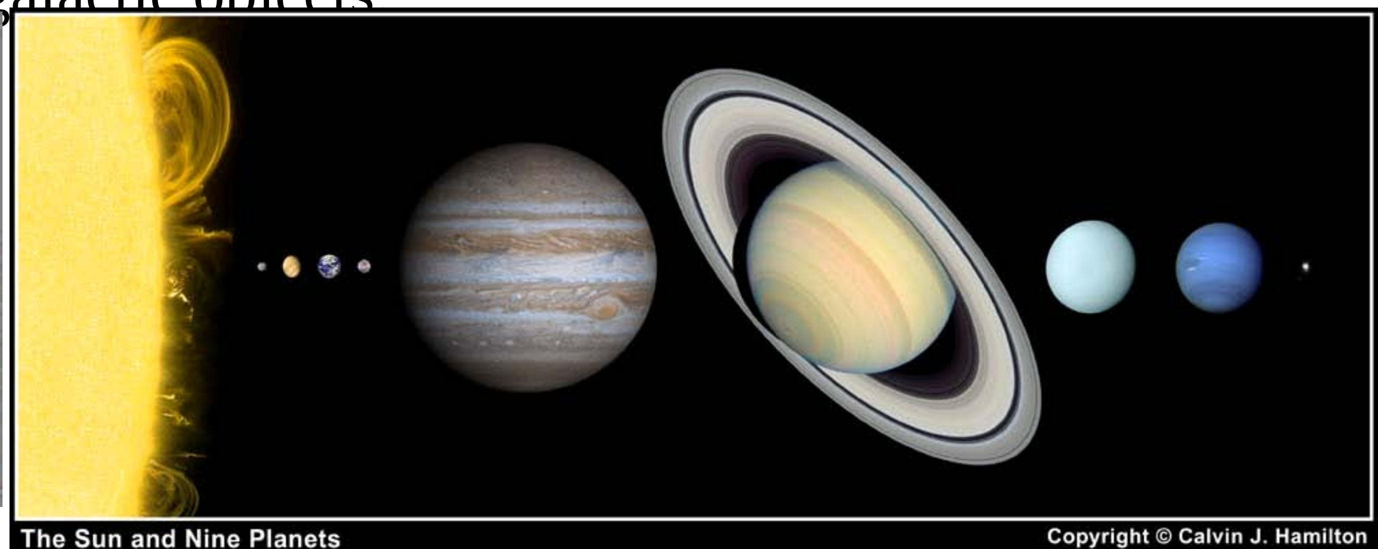


$$\dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2)\omega + R_{ij}u^i u^j = 0.$$

This equation was discovered by Raichaudhuri [31] (and independently by Landau), but is well known in literature as the *Raychaudhuri equation*, which plays a fundamental role in proving the Hawking–Penrose singularity theorems [4].

With Galileo’s telescope, the view of the Universe expanded rapidly from naked eye observations to cover the celestial beauty of Jupiter’s moons, Saturn’s rings and outer planets to the expanding Universe of Hubble that included, Galaxies, Clusters
and myriad extra galactic objects

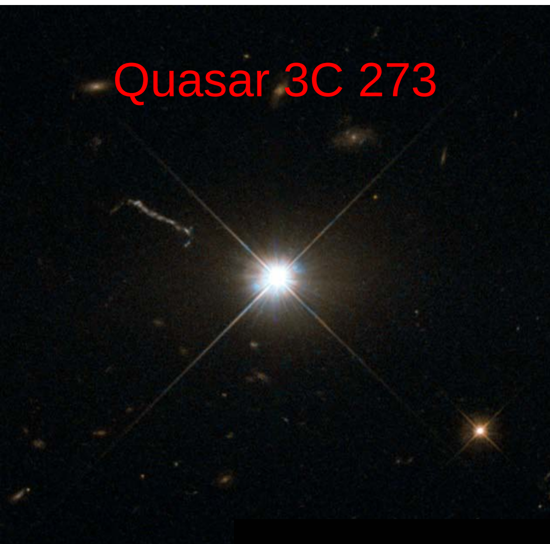


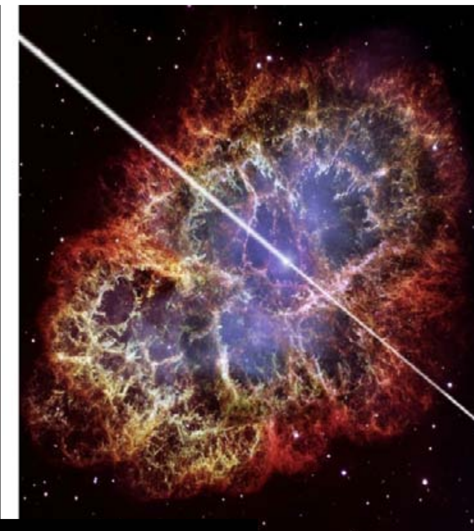


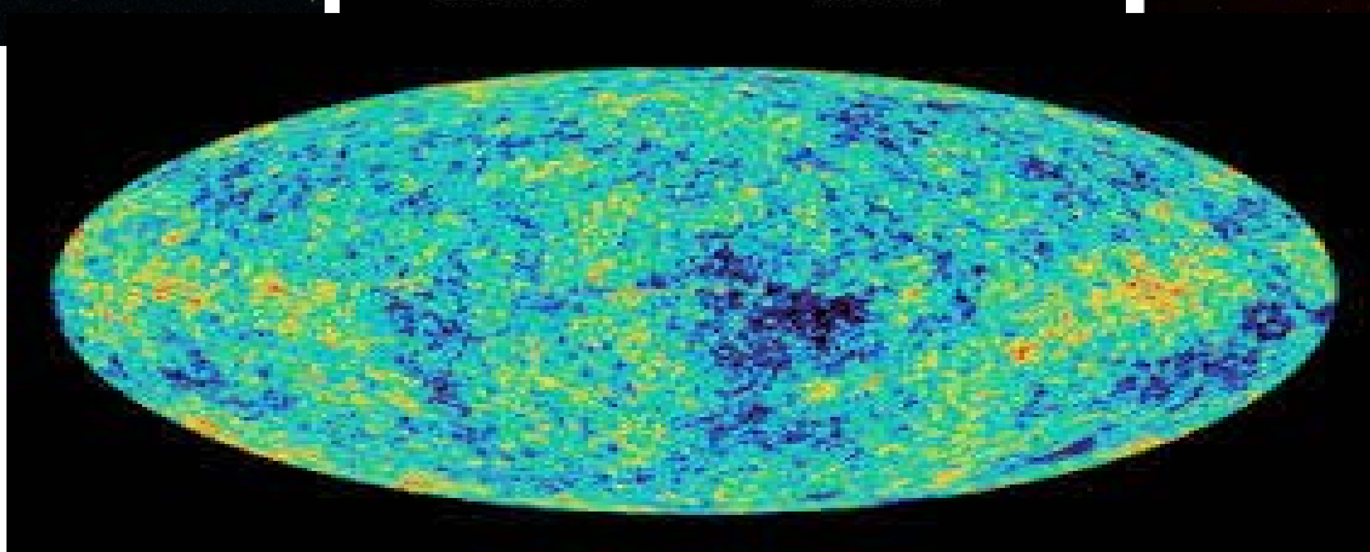

M31: The Andromeda Galaxy


Gemstones in the Southern Sky
NGC 290

In 1930, came the next revolution of the enigmatic Radio Universe, lead by Karl Jansky and G Reber, which opened very high energy cosmic sources like Quasars, Active Galactic Nucleii, Pulsars and more importantly, the most profound all encompassing ‘Cosmic microwave Background that indicated the beginning of our Universe.







Once the realisation came about emissions from cosmic sources in two different frequencies, the optical and the radio, it was a simple task to look for emissions in other frequencies, IR, UV, X-ray sources. Along with came the bonus of emissions of γ-rays , which Completed the electromagnetic spectrum of Universe being visible In the entire spectrum Radio waves to γ -rays

While it was known that the emission of radiation from most of these sources were all due to Electromagnetic processes, it was not clear upto 1960s, the source of energy for emissions from objects like Quasars and AGN s, till Hoyle and Fowler put forward the idea of Gravitational collapse of massive stars, within the framework of Einstein’s theory of General Relativity which explained Gravitation as the curvature of space-time.

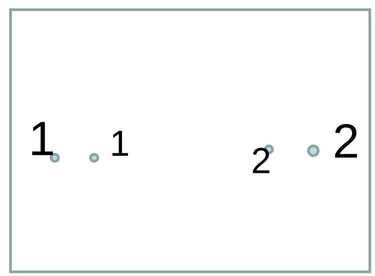
Of all the discoveries of the human mind, Einstein's theory of general relativity is considered to be the most beautiful creation. In fact, it is often said that the special relativity, which forms a strong basis of modern physics, along with quantum mechanics, was ripe to be discovered at the turn of the nineteenth century, and if not Einstein, Poincare or Lorentz would have developed the theory. On the other hand the general theory of relativity, which is the epitome of the world of symmetry, assigning freedom from the confines of coordinate systems (observers) to understand the most important of all the fundamental interactions-Gravity, is completely the work of one individual, arising out of thought experiments instead of laboratory experiments or observations that preceded all other discoveries in physics

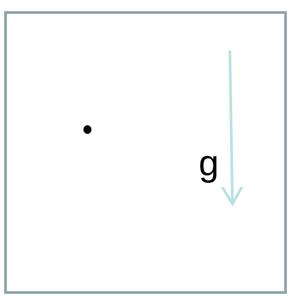
The most important features that lead Einstein from special to general relativity are-

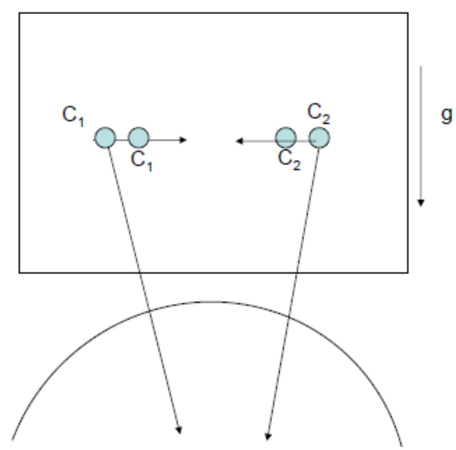
1.The equivalence of inertial and gravitational mass of any body, known as the principle of equivalence (demonstrated by Eotvos in 1889) and

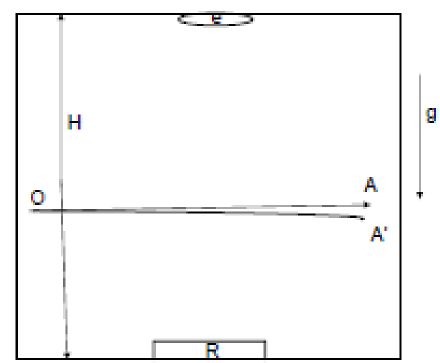
2. The effect of gravity on light –Einstein red shift (also bending of light by a gravitational field)

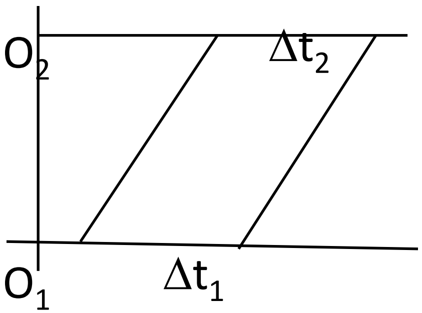
$$M_i\, a = M_g\, g \Longrightarrow a = g$$



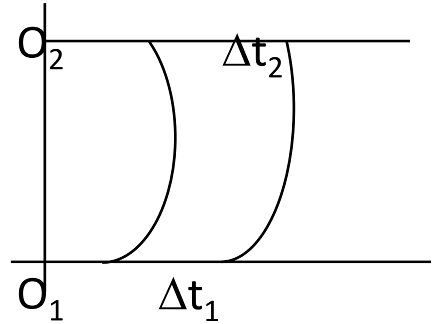






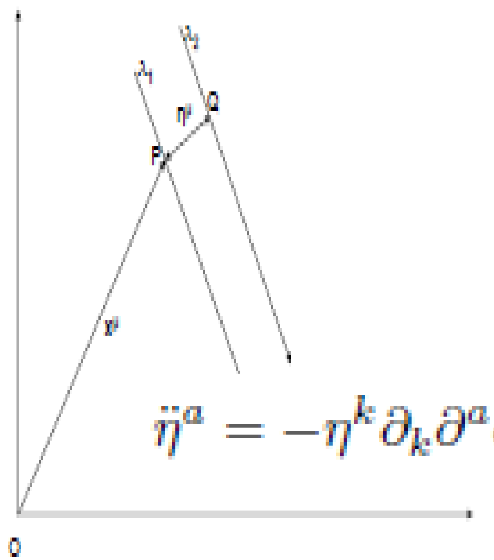


$f_1 > f_2$

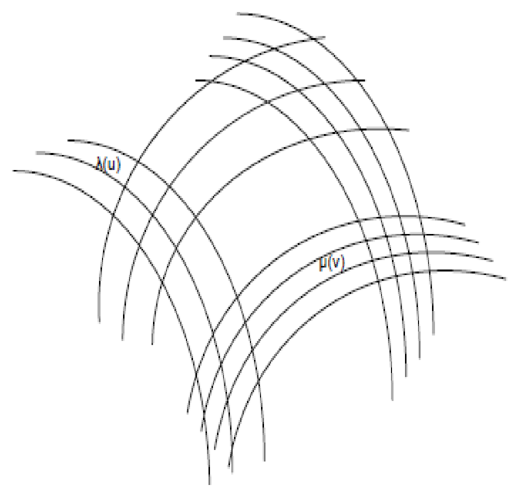


Einstein realized that the arena he wanted for general relativity, was the non Euclidean geometry, (non flat), which requirement was satisfied by the Riemannian Geometry, an extension of Gaussian curved geometry.

$$ds^2 = g_{ij}\, dx^i\, dx^j$$



$$\ddot{\eta}^a = -\eta^k \partial_k \partial^a \phi,$$



$$\frac{\delta^2 \eta^i}{\delta s^2} + R^i{}_{lkj} U^l \eta^j U^k = 0.$$

$$\frac{d^2 X^a}{ds^2} + R^a{}_{bcd} U^b X^c U^d = 0$$

The intrinsic curvature of a Riemannian space represented by the curvature tensor R_{hijk} represents the acceleration between two freely falling particles thus representing gravitation

The field equations of general relativity are $G_{ij} \equiv R_{ij} - ½ R g_{ij} = \kappa T_{ij}$ and they satisfy the identities $G^{ij}{}_{;j} = 0$, the Bianchi identities and the conservation laws $T^{ij}{}_{;j} = 0$.

As these equations are highly non linear and coupled set of p.d.es, it is difficult to look for general set of solutions. In fact Einstein himself in 1916,suggested that one could look for solutions in linearized approximation, assuming $g_{ij} = \eta_{ij} + h_{ij}$

Outside the sources, $T_{ij} = 0$, makes the field equations reduce to $R_{ij} = 0$,

$$\Box h_{ij} + h^k{}_{k,ij} - h^k{}_{i,jk} - h^k{}_{j,ik} = 0,$$

As there still exists general covariance, one chooses a particular gauge, known as Lorentz gauge or the harmonic gauge $g^{jk} \Gamma^i{}_{jk} = 0$, which reduces the equation to the simple wave equation

$$\Box h_{ij} = 0.$$

$$h_{ij} = A_{ij} e^{ik_l x^l} + A^*_{ij} e^{-ik_l x^l},$$

with A and A^* representing the complex amplitudes and k^l the wave covector, satisfying the orthogonality relation, $\eta_{ij} k^i k^j = 0$.

The gauge condition, yields four constraints on the ten complex amplitudes, given by the relation

$$A_{ij} k^j = \frac{1}{2} A^j_j k_i.$$

$$A_{ij} k^j = 0, \quad A_{ij} u^j = 0, \quad A^i_i = 0,$$

Eight constraints on ten amplitudes

In terms of metric potentials, this means, $h_{i0} = 0, \quad h_{a\ ,j}^{\ j} = 0, \quad h_i^i = 0.$

As there are only two degrees of freedom, this means physically that the waves have only two degrees of polarisation

Thus for a plane gravitational wave propagating along the Z-direction, in a Cartesian system, the solution may be written explicitly as

$$h^{TT}{}_{XX} = -h^{TT}{}_{YY} = \mathscr{R}\{a_+ e^{[-i\omega(t-z)]}\}$$

$a_+ = A_{11} = - A_{22}$

$$h^{TT}{}_{XY} = h^{TT}{}_{YX} = \mathscr{R}\{a_\times e^{[-i\omega(t-z)]}\}$$

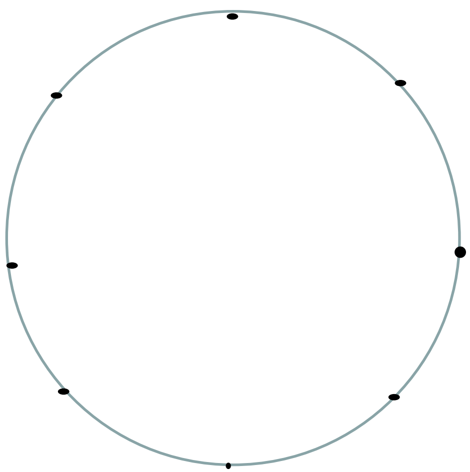
$a_x = A_{12} = A_{21}$

$$ds^2 = dt^2 - (1 - h_{XX})dx^2 - (1 - h_{YY})dy^2 + 2h_{XY}dxdy - dz^2$$

$$(a)\ R^x{}_{0x0} = -\frac{1}{2}h^{TT}{}_{XX,00},$$

$$(b)\ R^y{}_{0y0} = \frac{1}{2}h^{TT}{}_{YY,00},$$

$$(c)\ R^x{}_{0y0} = R^y{}_{0x0} = -\frac{1}{2}h^{TT}{}_{XY,00}.$$



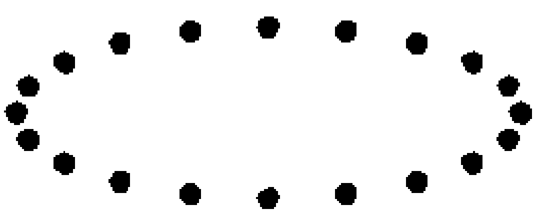
In a commoving frame, $u^i = (1,0,0,0)$ and the deviation vector

$\eta^i = (0, \varepsilon, 0, 0)$

$$(i)\ \frac{\partial^2 \eta^x}{\partial t^2} = \frac{1}{2} h^{TT}{}_{XX,00}\ \varepsilon,$$

$$(ii)\ \frac{\partial^2 \eta^y}{\partial t^2} = \frac{1}{2} h^{TT}{}_{XY,00}\ \varepsilon.$$

If $h_{xy} = 0$, & $h_{xx} = -h_{yy} \neq 0$,

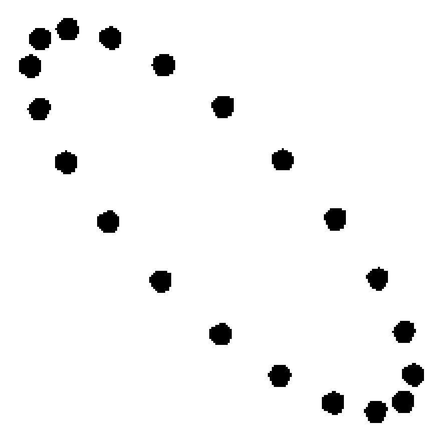


$\eta^i = (0, 0, \varepsilon, 0)$

$$(iii)\ \frac{\partial^2 \eta^x}{\partial t^2} = \frac{1}{2} h^{TT}{}_{XY,00}\ \varepsilon,$$

$$(iv)\ \frac{\partial^2 \eta^y}{\partial t^2} = -\frac{1}{2} h^{TT}{}_{XX,00}\ \varepsilon.$$

If $h_{xy} \neq 0$, & $h_{xx} = -h_{yy} = 0$,



Do these waves carry Energy and angular momentum?

The conservation laws may be expressed as
$$\frac{1}{\sqrt{-g}}[\frac{\partial(T_i^j\sqrt{-g})}{\partial x^j}] - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}T^{jk} = 0,$$

Rewriting this in the form
$$\frac{\partial}{\partial x^j}[(-g)\,(T_i^j + t_i^j)] = 0,$$

The field energy part also known as Landau- Lifshitz pseudo tensor t^{ik} is defined

$$(-g)[(R^{ik} - \frac{1}{2}Rg^{ik}) + t^{ik}] = \Psi^{ikl}{}_{,l}. \qquad \Psi^{ikl} = \sqrt{-g}\,\delta^i_p\,\{g(g^{kp}g^{lm} - g^{km}g^{lp})\}_{,m}.$$

In the case of linearized gravity, with perturbations over a flat background, for the short wave approximation as defined by $(\lambda/\mathcal{R})^2 \ll 1, a \ll 1,$

the effective stress tensor averaged over several wave lengths is given by

$$t_{ij} = \frac{1}{8\pi}\{< R_{ij}(h^2) > -\frac{1}{2}g^B{}_{ij} < R(h^2) >\},$$

Which in the TT gauge gives the Issacson stress tensor
$$< t_{ij} >= \frac{1}{32\pi} < h^{kl}{}_{,i}\, h_{kl,j} > .$$

$$t_{00} = t_{zz} = -t_{0z} = \frac{1}{32\pi}\omega^2\,(|a_+|^2 + |a_\times|^2).$$

Thus the gravitational waves on the flat background,follow null geodesics $(\eta^{ij} k_i k_j = 0)$ possess two independent states of polarisation and carry energy proportional to the sum of the squares of the amplitudes.

Field equations with the source then written as (in the harmonic gauge), along with integrability conditions are $\Box \bar{h}_{ij} = -2\kappa \tau_{ij}, \ (\tau_i^j)_{,j} = 0.$

and the solution may be written in terms of retarded Green's function

$$\bar{h}_{ij} = 4 \int \{ [\frac{\tau_{ij}(x', t - |x - x'|)]}{|x - x'|} \} \, d^3 x'$$

From the conservation law,

$$\tau^{ij}_{,j} = 0, \qquad (a)\, \tau^{ab}{}_{,b} + \tau^{a0}{}_{,0} = 0$$
$$(b)\, \tau^{0b}{}_{,b} + \tau^{00}{}_{,0} = 0.$$

$$\int \tau^{ab} dV = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int \rho(r', t) x^a x^b dV = \frac{1}{2} \ddot{I}^{ab},$$

I^{ab} is the second moment of the mass distribution at the source, related to the Moment of Inertia tensor,

$$\mathscr{I}^{ab} = \int \rho(r^2\delta^{ab} - x^a x^b)dV = (\delta^{ab} I^c_c - I^{ab}),$$

With this, one can write the final approximate solution to the field equation to be

$$\bar{h}_{ij} = \frac{-2\Omega^2}{r} I_{ij} e^{[i\Omega(r-t)]},$$

Ω being the frequency.- the well known Quadrupole formula.

The metric potentials are given by

$$h_{Zi} = 0, (i = 0, 1, 2, 3); \quad h_{XY} = -\frac{2\Omega^2}{3r} Q_{XY} e^{[i\Omega(r-t)]}$$

$$h_{XX} = -h_{YY} = -\frac{\Omega^2}{3r}(Q_{XX} - Q_{YY})e^{[i\Omega(r-t)]},$$

While the energy flux Carried along the direction Of propagation is

$$t^{z0} = (\frac{G}{36\pi r^2 c^5})[(\frac{\dddot{Q}_{XX} - \dddot{Q}_{YY}}{2})^2 + (\dddot{Q}_{XY})^2].$$

The intensity of radiation of a given polarisation into a given solid angle dΣ, is

$$dI = \frac{1}{72\pi}(\dddot{Q}_{ab} e^{ab})^2 d\Sigma,$$

By averaging $dI/d\Sigma$ over all directions and multiplying by 4π , one can find the energy and angular momentum loss of the system per unit time

$$\frac{dE}{dt} = -\left(\frac{G}{45c^5}\right) < \dddot{Q}_{ab}\dddot{Q}^{ab} >$$

$$\frac{dJ_k}{dt} = -\left(\frac{2G}{45c^2}\right)\varepsilon_{klm} < \ddot{Q}^{la}\dddot{Q}^{m}_{a} > .$$

The luminosity expressed in terms of the local stress energy in the T-T gauge, is given by

$$L_{gw} = \frac{1}{5}\left(\Sigma_{j,k}(\dddot{Q}_{jk})^2 - \frac{1}{3}(\dddot{Q})^2\right),$$

$Q = Q^{ij}Q_{ij}$ representing the trace.

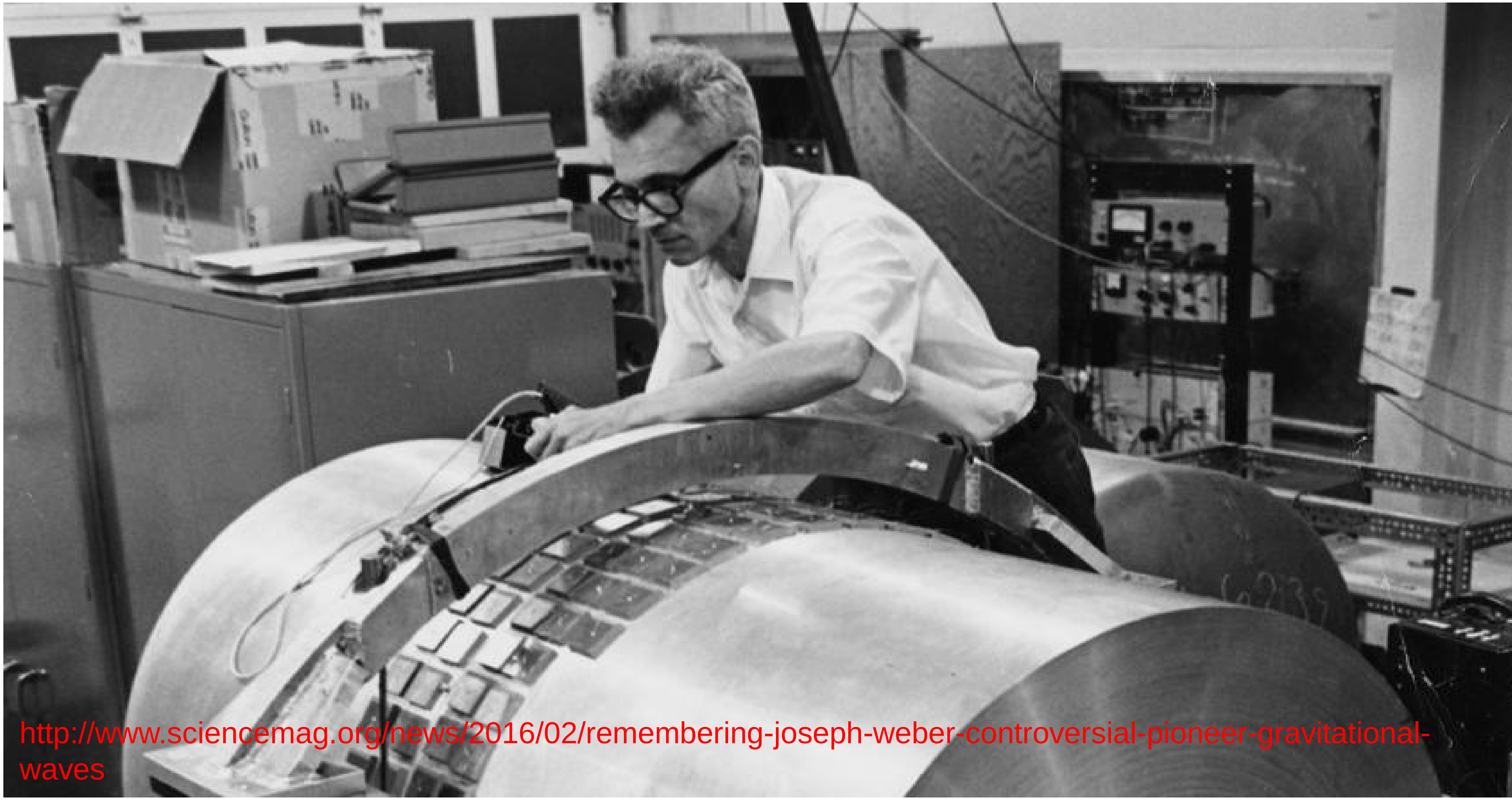
The amplitude and the frequency in Lorentz gauge are given by

$$h_{ab} = \frac{2d^2Q_{ab}}{rdt^2}, Q^{ab} = \int \rho x^a x^b d^3x \qquad f_0 = \omega_0/2\pi = \sqrt{G\overline{\rho}/4\pi}$$

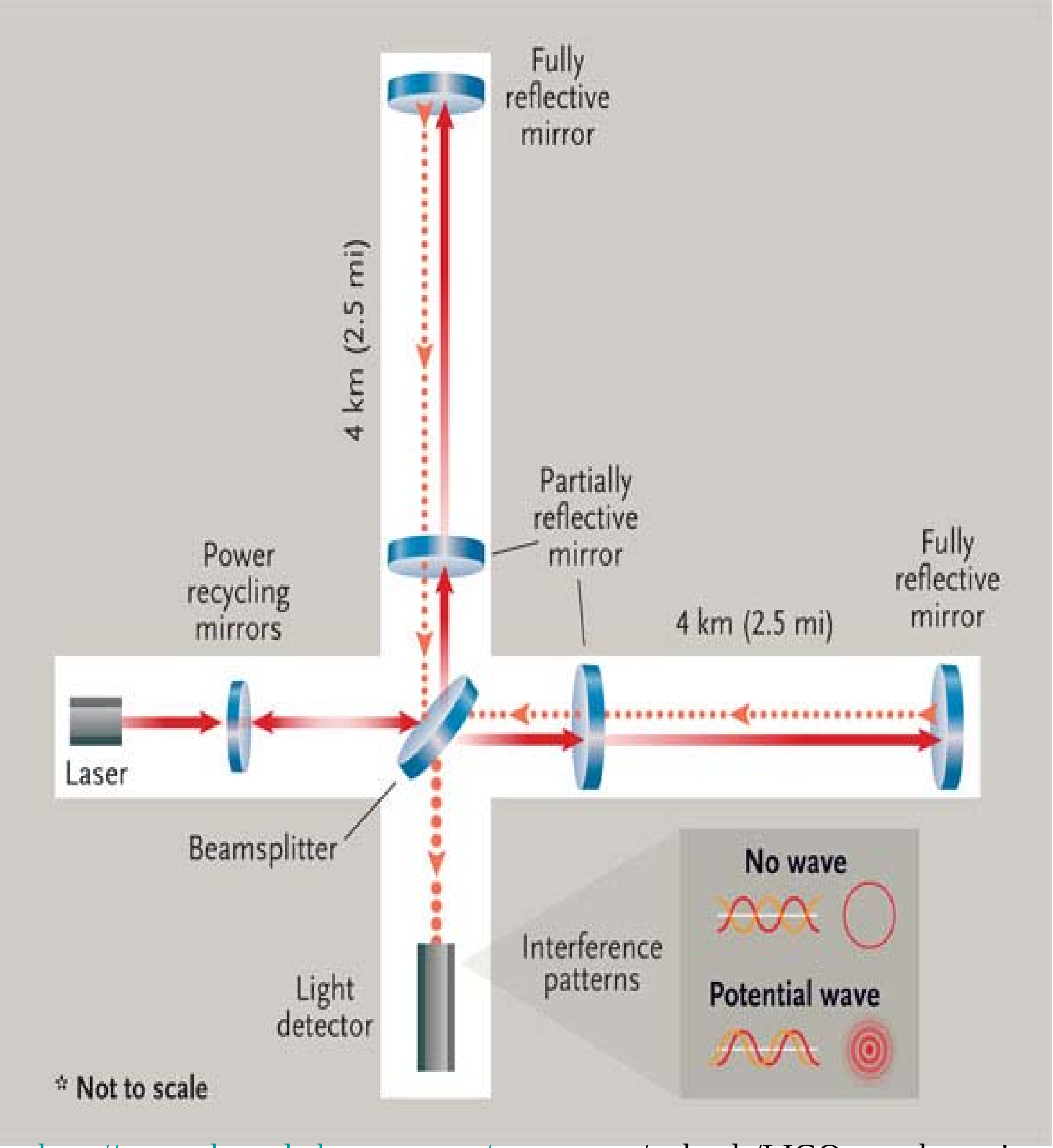
Detection of Gravitational waves

As one can see from the expression for the energy it goes as $c^{(-5)}$ which is very small, Indicating that one would require an extremely sensitive detector, to separate the signal from the noise. Two types of detectors were used, bar mode detector and beam mode detector

J.Weber – resonant bar detector - gravitational waves (gwvs) would induce mechanical vibrations which are converted to electrical signals and analysed.


http://www.sciencemag.org/news/2016/02/remembering-joseph-weber-controversial-pioneer-gravitational-waves

Joe Weber, working on one of his resonant bar detector in early 1960s, a suspended homogeneous metal bar, on which an impinging gravitational wave would excite mechanical vibrations, that could be transferred to electromagnetic signals by piezoelectric transducers which can be amplified and recorded. Coincident signals recorded at two or more such detectors far from each other, will show the arrival of gravitational waves.



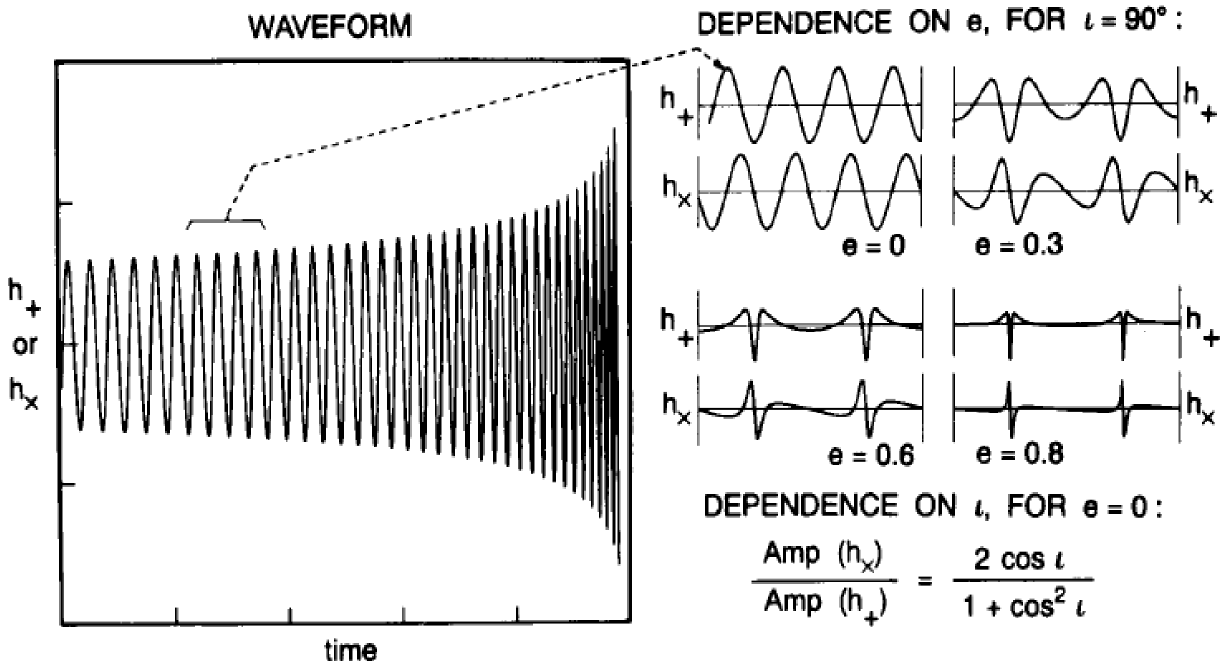
http://www.skyandtelescope.com/wp content/uploads/LIGO schematic

Four masses hung from vibration free system with their separation being monitored by a highly sophisticated optical system. The two arms are of almost of equal length such that $\Delta L(t) = L_1 - L_2$ is directly proportional to the output of the photodiode. When a gravitational wave having frequency higher than pendulum's natural frequency of about 1 Hz, passes through, the relative acceleration induced by the wave pushes the masses, causing ΔL to change. Depending upon the polarization of the impinging wave the interferometer's output would be a linear combination of the two fields

$$\frac{\Delta L(t)}{L} = F_+ h_+(t) + F_\times h_\times(t) \equiv h(t)$$

where Fs are of order unity having quadrupolar dependence upon the direction and orientation to the source and h(t) the strain of the wave, with their time evolution giving the wave form.

A typical wave form coming from an inspiralling binary system , computed using Newtonian gravity for the orbit evolution and qudrupole-moment approximation for wave generation would look like what is shown in the figure, where one finds that as the inspiralling gets closer, amplitude increases and one has an upward sweeping frequency of the wave form



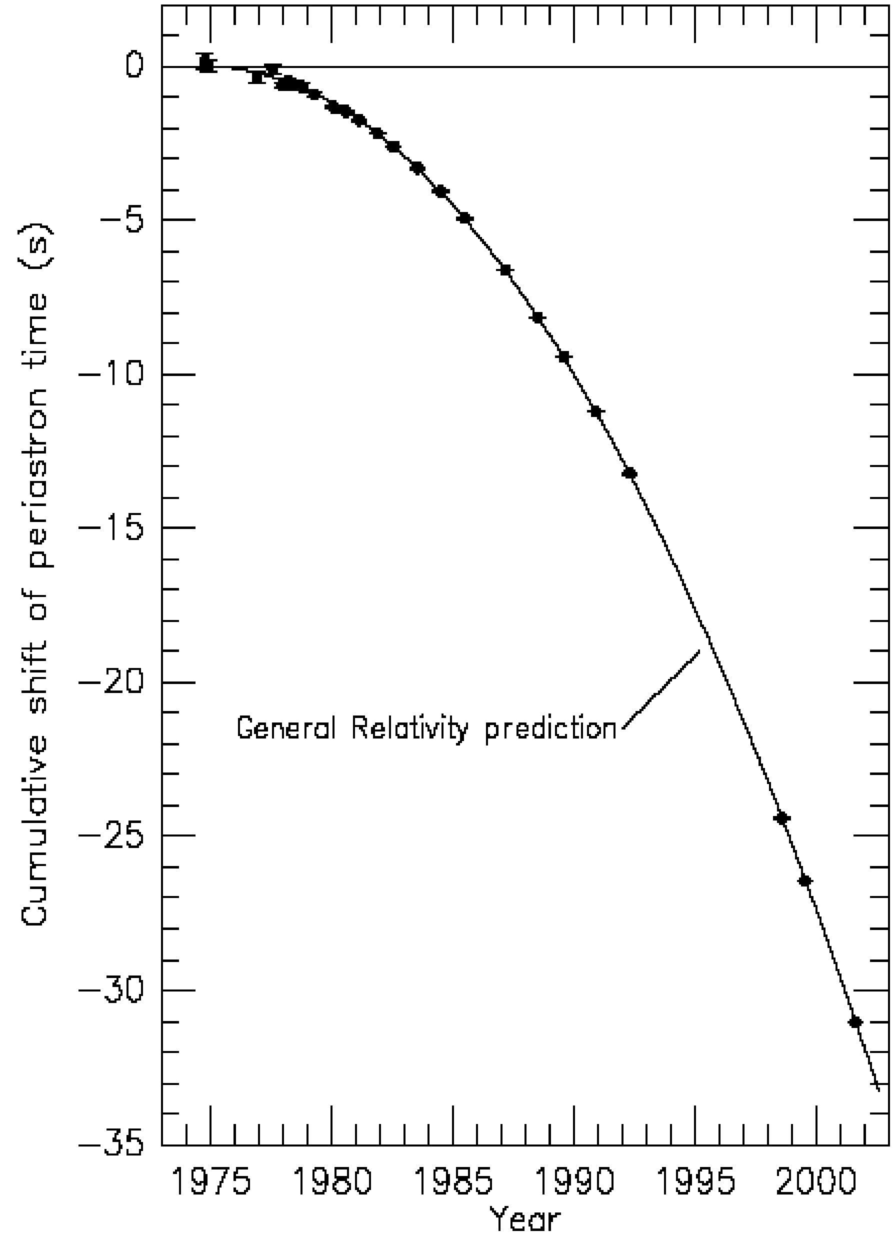
(often called a "chirp") with amplitude ratio for the two polarization going as $\text{amp } h_+ / \text{amp } h_X = 2 \cos i / (1 + \cos^2 i)$, i being the inclination of the binary orbit to the observer's line of sight.

The orbital eccentricity determines the harmonic content of the wave. Thus one can, by measuring the two amplitudes, frequency sweep, and the harmonic content of the inspiralling waves, determine the source's distance, chirp mass, inclination and eccentricity.

(K S Thorne 97) (aeXive gr-qc / 9704042).

The first successful indirect detection of gws was from the H-T binary pulsar, which was discovered in 1974, and by 1982, the continuous recording of the pulse arrival time had confirmed the shrinking of the binary orbit due to emission of gravitational waves.

By 1982, the pulsar was arriving at its periastron more than 1 sec. earlier than would have been expected if the orbit had remained constant since 1974.Data in the first decade after the discovery showed a decrease in the orbital period of about 76 millionths of a second/year.



$$(\dot{P}_b)^{obs}/(\dot{P}_b)^{GR} = 1.0081 \pm 0.0022(gal) \pm 0.0076(obs)$$

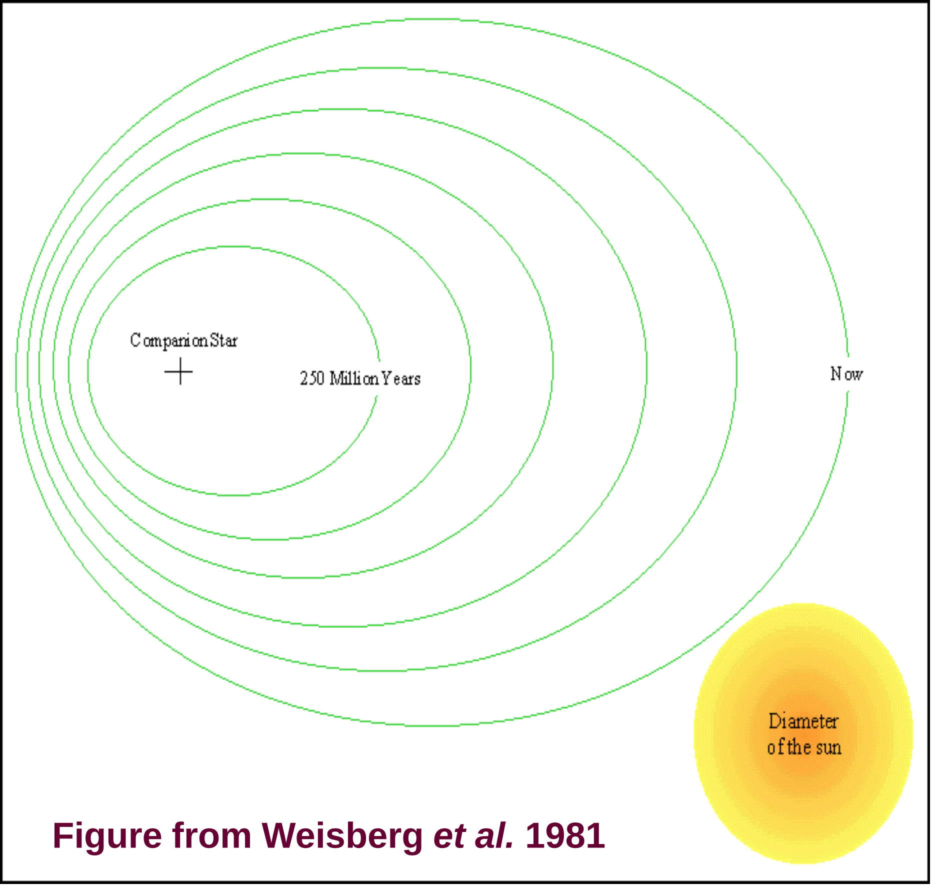


Figure from Weisberg *et al.* 1981

As the orbits shrink, the binary components inspiral and coalesce giving rise to a catastrophic event, which can produce a huge flux of gravitational radiation.

The H-T binary will coalesce in about 250 million years.

Coalescence of massive binaries are among the strongest sources of gravitational wave signals with energetics measurable by the ground observatories.
Black hole mergers are the best candidates.

This has three phases, called, inspiral, merger and ringdown

During the inspiral phase, the orbits of the binary components get circularised due to emission of gwvs, and the components spiral together in quasi circular orbits, as the orbital time scales would be much shorter than the time scales on which the orbital parameters change. Weak field approximation is valid only before the merger phase, and there after the strong field general relativistic effects come into force, when one has to adopt numerical methods and computer simulations for building the templates for signal matching.

The final phase is the ring down phase, when the merged black holes could form a highly distorted single Black hole, which may finally settle down to a Kerr black hole , smoothening itself after shedding all the non axisymmetric modes by emitting gravitational waves in the form of Quasi normal modes characterised by Mass and Spin of the black hole only.

Working out the dynamics of this final phase is one of the most challenging problems of numerical relativity, as it deals with the black hole perturbation theory.

Perturbations of Schwarzschild blackhole –initiated by C.V.Vishveswara 1970, further extended by F.Zerilli. Perturbatons of Kerr blackhole – studied by S.Teukolsky

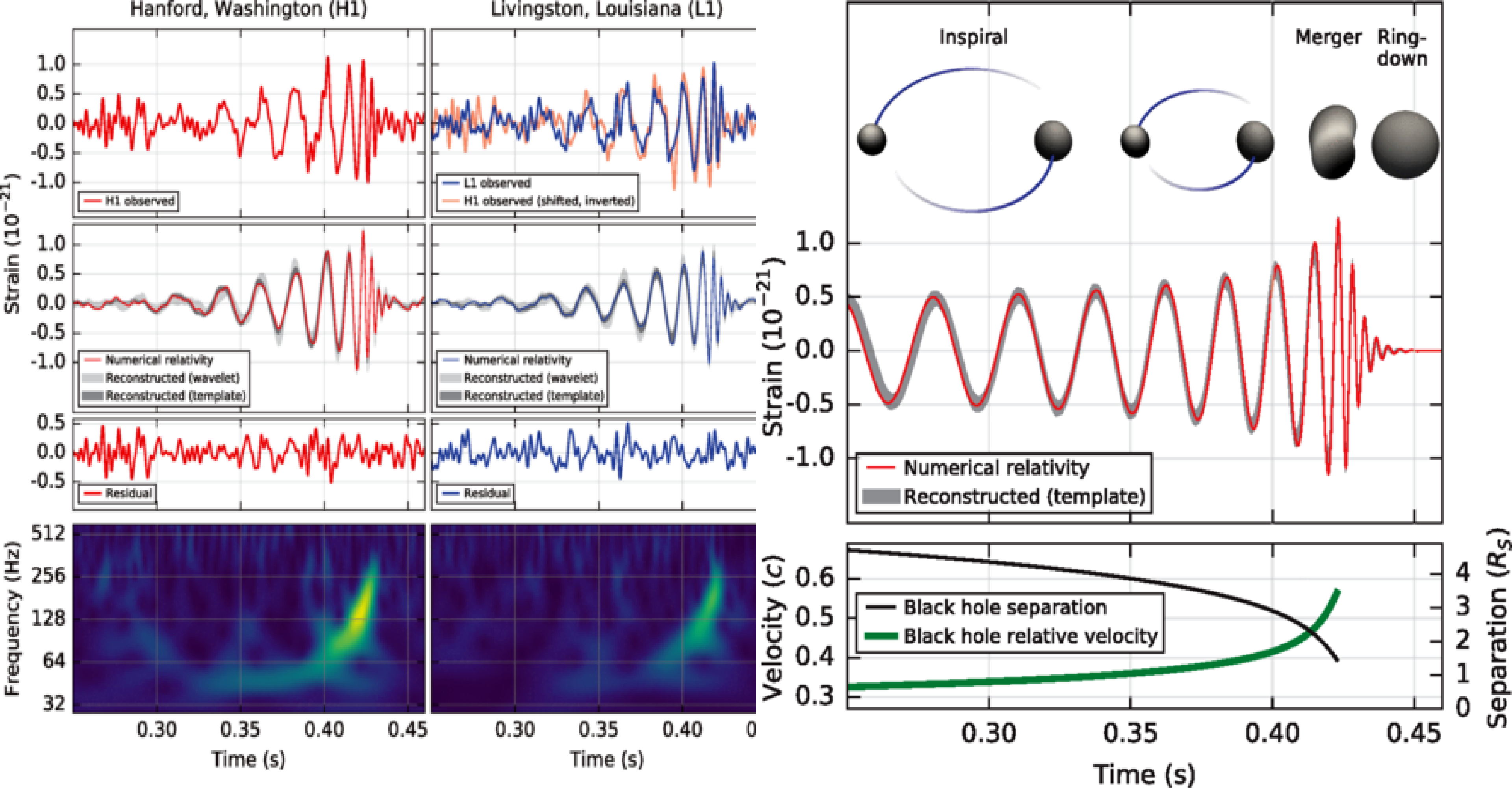
Full formalism of blackhole perturbations Chandrasekhar et al - 70s & 80s
S. Chandrasekhar "The Mathematical Theory of Blackholes" 1983.

C.M.Will – post Newtonian approximation and PPN schemes

T.Damour & B.R.Iyer on approximation methods

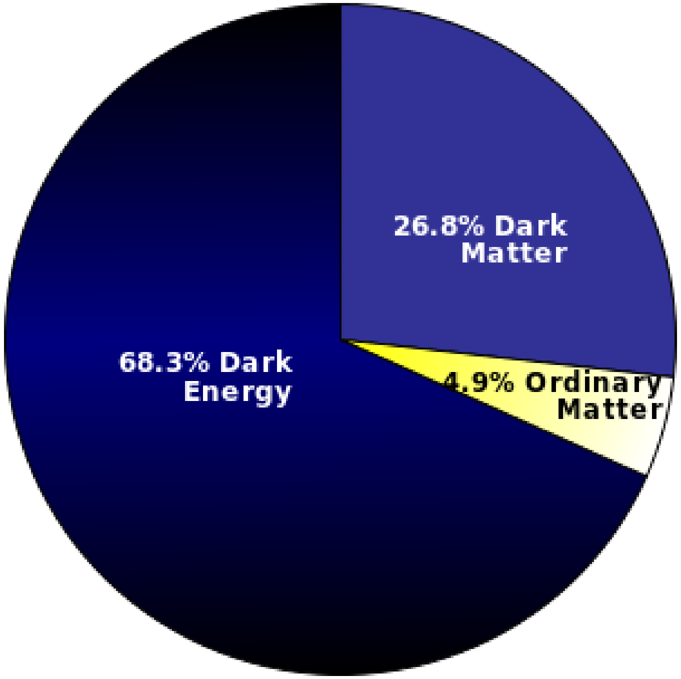
A big step in Numerical Relativity by Pretorius et al.

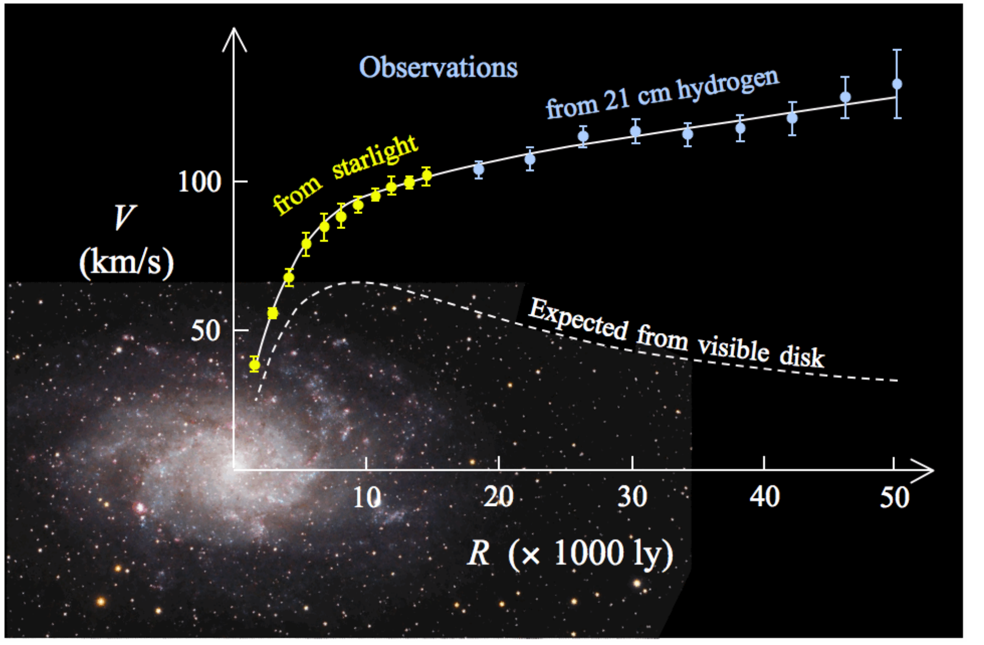
Template construction for match filtering techniques – B.S.Satyaprakash



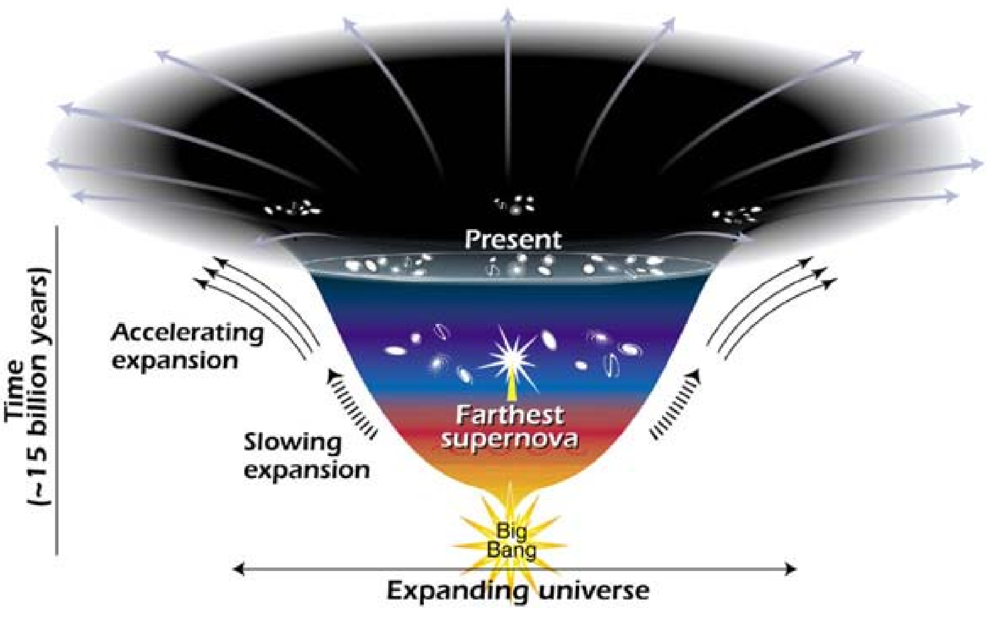
The gravitational-wave event GW150914 observed by the LIGO Hanford (H1, left column panels) and Livingston (L1, right column panels) detectors. Times are shown relative to September 14, 2015 at 09:50:45 UTC. For visualization, all time series are filtered with a 35–350 Hz band pass filter to suppress large fluctuations outside the detectors' most sensitive frequency band, and band-reject filters to remove the strong instrumental spectral lines seen in the Fig. 3 spectra.

Masses of the bhs 29 and 36 solar mass, distance 410 +/- 135 Mpcs.





Dark energy is mysterious, and its discovery in the 1990s was a complete shock to scientists. Physicists had assumed that the attractive force of gravity would slow down the expansion of the universe over time. But when two independent teams tried to measure the rate of deceleration, they found that the expansion was actually speeding up. Scientists now think that the accelerated expansion of the universe is driven by a kind of repulsive force generated by quantum fluctuations in otherwise "empty" space.



Propagation of gravitational waves through matter Ehlers and Prasanna - 1986 , 93

Starting with the perturbations on a non flat background $g_{ij} = (g_{ij})_B + \hat{g}_{ij}$

one can set up the equations for perturbations, for the case of perfect fluid distribution as given by

$$P^{ij}{}_{ab}\,\hat{g}_{ij} \equiv [(2h^i_{(a}h^c_{b)}\nabla^j\nabla^c - h^i_a h^j_b \nabla^2 - g^{ij}h^c_a h^d_b \nabla_c \nabla_d) + \alpha h_{ab}(g^{ij}\nabla^2_u + 2\nabla^{(i}u^{j)}\nabla_u + 4(\nabla_d u^i \nabla^{[d}u^{j]})) - (\rho - p + 2\Lambda)h^i_a h^j_b](\hat{g}_{ij}) = 0.$$

The zeroth order equation gives the dispersion relation

$$(g^{ab}l_a l_b)^2[(u^a u^b - c_s^2 h^{ab})l_a l_b](u^a l_a)^6 = 0.$$

(i) the *gravitational wave* mode, given by the Hamiltonian $H = \frac{1}{2}g^{ab}l_a l_b$ propagating along the null geodesics having the tangent vector $T^a = l^a$,

(ii) the *sound wave* mode given by $H = \frac{1}{2}[(c_s^2 h^{ab} - u^a u^b)]$, propagating along the sound rays, with tangent $T^a = \omega(\frac{c_s k^a}{k} + u^a)$, and

(iii) the *matter* mode given by $H = u^a l_a$, moving along matter rays, $T^a = u^a$, From the general formalism, one can find that while the zeroth order equations of the Hierarchy of equations $L_0 V_0 = 0$, provide the dispersion relation, the first order equations $L_0 V_1 + L_1 V_0 = 0$, gives the transport equations for the amplitudes.

For the primary amplitudes, one finds from the first order equation, the set of ordinary differential equations

$$(\nabla_l + \frac{\theta}{2}) \begin{pmatrix} a_+^{(0)} \\ a_\times^{(0)} \end{pmatrix} = 0. \qquad \boldsymbol{\theta = \nabla^a l_a}$$

This implies that the change of the complex vector $(a_+, a_\times)$ along a ray consists of a rescaling by a positive factor proportional to the square root of the cross–sectional area of a small bundle of rays, just as in the case of gravitational waves in *vacua*. The transport preserves linear, circular, elliptic polarisation, helicity and ellipticity. Further it also implies that the Issacson stress tensor (defined in vacuum)

$$\hat{T}^{ab} = \frac{1}{4\pi}(|a_+|^2 + |a_\times|^2) l^a l^b,$$

which represents the effective energy momentum tensor of the wave, is conserved, $\nabla_a \hat{T}^{ab} = 0$.

The transport equation for the first order primary amplitudes is then given by,

$$(\nabla_l + \theta/2)(a_+^{(1)}) = \frac{1}{2}(\rho - p + 2\Lambda)((a_+^{(0)}) + \\ + \frac{1}{2}e^{ij}\{[2\nabla^c\nabla_i + \nabla^i\nabla_c)\delta_j^d - \delta_i^c\delta_j^d\nabla^2 - h^{cd}\nabla_j\nabla_i](v_{0cd})\}$$

Using the field equations one can see that for a conformally flat space-time , $C_{hijk} = 0$,

$$(\nabla_l + \theta/2)(a_+^{(1)}) + \frac{R}{3}((a_+^{(0)}) = \frac{1}{2}e_+^{ij}[4\nabla_i\nabla^c\delta_j^d - \delta_i^c\delta_j^d\nabla^2 - h^{cd}\nabla_j\nabla_i)](v_{0cd}).$$

This equation exhibits the possibility of the background curvature R and the nonlinear derivatives of the primary amplitudes v_0 influencing the transport of v_1, the correction to the primary amplitude

Instead of a perfect fluid, if one had viscous fluid $T_{ij} = (\rho + p)u_iu_j + pg_{ij} - 2\eta\sigma_{ij} - \zeta\theta h_{ij}$,

A.R.Prasanna Phys.Letts., A 257, p 10, (1999)

The perturbed field equations $\hat{R}_{ij} = \kappa[\hat{T}_{ij} - \frac{1}{2}(g_{ij}\hat{T} + \hat{g}_{ij}T)]$

along with the gauge condition $\hat{g}_{ab}U^b = 0$,

and the fact that the unperturbed stream lines of the fluid are geodesics

$$H^{ij}{}_{ab}\hat{R}_{ij} = (\kappa/2)[\hat{g}_{ab}(\rho - p + \zeta\theta) + h_{ab}(4\zeta\hat{\theta}/(1+3c_s^2)) - 4\eta\hat{\sigma}_{ab}],$$

with

$$H^{ij}{}_{ab} = h^i_a h^j_b - \alpha h_{ab} u^i u^j,$$

$$\hat{\theta} = \hat{u}^k_{,k} + \frac{g^{ka}}{2}(g_{ka,b}\hat{u}^b + \hat{g}_{ka,b}u^b) + \frac{\hat{g}^{ka}}{2} g_{ka,b}u^b,$$

$$\nabla_j \hat{u}_i = \hat{u}_{i,j} - \frac{u^b}{2}(\hat{g}_{ib,j} + \hat{g}_{jb,i} - \hat{g}_{ij,b}) + \{ij, b\}(u_k \hat{g}^{kb} + \hat{u}^b).$$

Using now the eikonal approximation suitable for the high frequency limit, one gets at the zeroth order the dispersion relation, $l^4\omega^6[\omega^2 - c_s^2 k^2] = 0,$ having the three modes as earlier

Going to the next higher order, one finds the transport equation for the amplitudes

$$[l^i\nabla_i + \frac{1}{2}\nabla_i l^i + \kappa\eta\omega]e_+^{ab} f_{ab} = 0 \Rightarrow [\nabla_l + \frac{1}{2}\nabla_i l^i + \kappa\eta\omega)\begin{pmatrix} a_+^{(0)} \\ a_\times^{(0)} \end{pmatrix} = 0,$$

Which in terms of the total amplitude reads $(D + \nabla^i l_i)A^2 = -2\kappa\eta\omega A^2,$

LIGO research is carried out by the LIGO Scientific Collaboration (LSC), a group of more than 1000 scientists from universities around the United States and in 14 other countries. More than 90 universities and research institutes in the LSC develop detector technology and analyze data; approximately **250 students** are strong contributing members of the collaboration. The LSC detector network includes the LIGO interferometers and the GEO600 detector. The GEO team includes scientists at the Max Planck Institute for Gravitational Physics (Albert Einstein Institute, AEI), Leibniz Universität Hannover, along with partners at the University of Glasgow, Cardiff University, the University of Birmingham, other universities in the United Kingdom, and the University of the Balearic Islands in Spain.

LIGO-India is a planned advanced gravitational-wave observatory to be located in India as part of the worldwide network. The project recently received the in-principle approval from the Indian government. LIGO-India is planned as a collaborative project between a consortium of Indian research institutions and the LIGO Laboratory in the USA, along with its international partners Australia, Germany and the UK.