Towards singularities................

Narayan Banerjee

IISER Kolkata

Vaidya-Raychaudhuri Endowment Award Lecture
IIT Guwahati

Singularities of Indian Relativity Community

Prahlad Chunnilal Vaidya (1917-2010)

Vaidya metric – The exterior spacetime of a shining star.

Amal Kumar Raychaudhuri (1923-2005)

Raychaudhuri equation - The main instrument in detecting the existence of singularities.

Raychaudhuri Equation

AKR in 1950's

His main concern: Is the cosmological singularity generic?

His work: Search for non-singular solutions.

Hubble's observation, 1929

Galaxies are all moving away from each other

Look back in time – they were closer together

Look further back – they were even closer together

At some time in the past, all were together, in a zero volume!

SINGULARITY

Possibility of escape from singularity

Escape from this embarassment ?

Two options:

1. The zero of volume is reached at an infinite past
2. There was a minimum of the volume

Theoretical Model, 1922

1. The Universe is spatially homogeneous and isotropic

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} - r^2(d\theta^2 + \sin^2 d\phi^2)]$$

2. The matter distribution in the unverse can be approximated as a fluid

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu - pg_{\mu\nu}$$

3. General Relativity is the correct theory of gravity

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

Theoretical Model: Solution

Einsteuin equations

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = 8\pi G\rho,$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi Gp$$

Solutions for a standard barotropic fluid ($p = \alpha\rho, 0 \leq \alpha \leq 1$):

$$a \sim (t - t_0)^n$$

where $0 \leq n \leq 1$.

So there is no escape from the singularity!!

General situation

Look at it in a bit more general way:

Combine Einstein equations to get

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p)$$

For any matter distribution, this is the key equation.

$(\rho + 3p) \geq 0 \rightarrow$ Strong energy condition.

$\ddot{a}$ is negative definite, indicating

There is no minimum for the scale factor !!

The big question

The question that is asked:

Is this singularity an artefact of the high degree of symmetry?

In what ways the departure from this high degree of symmetry can appear for a general kind of spacetime?

Does this departure from symmetry have any role to play in avoiding the singularity?

Generalization

One is actually interested in the motion of a particle, meaning the velocity vector.

There are actually three different ways in which one can lose the high degree of symmetry.

All the three routes can be constructed from the

velocity vector

in terms of combinations of its derivatives.

(Derivatives in the tensorial way!)

Basic ingredients

v^{μ} is the timelike velocity vector.

The covariant derivative of $v^{\mu} \rightarrow v^{\mu}_{;\nu} \rightarrow$ a two index object.

The "trace", $v^{\mu}_{;\mu}$, labelled as θ,

called the "expansion scalar"

$\rightarrow$ a measure of the fractional rate of increase of the volume against proper time.

The tracefree part forms "shear" and "rotation".

Basic ingredients

The other important quantity is the projection of the derivative $v^{\mu}_{;\nu}$ along the time like direction v^{μ},

$$v^{\mu}_{;\nu} v^{\nu}$$

This is called the acceleration vector, a^{μ}

Shear

Shear appears due to the departure from isotropy.

If the universe expands **NOT** in the same way in all spatial directions

there is a distortion of shape.

Shear gives a measure of this distortion.

$$\sigma_{\alpha\beta} = \frac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha}) - \frac{1}{2}(a_\alpha v_\beta + a_\beta v_\alpha) - \frac{1}{3}\theta(g_{\alpha\beta} - v_\alpha v_\beta).$$

This is a tracefree symmetric tensor, and is orthogonal to the velocity vector.

The scalar measure of the shear is defined as

$$\sigma^2 = \frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta}$$

Vorticity

This appears if there is a rotation in the spacetime.

The rotation is given by the curl of the velocity vector.

Thus it is the **antisymmetric** combination of the derivatives of the velocity vector.

This tensor is orthogonal to the velocity vector.

$$\omega_{\alpha\beta} = \frac{1}{2}(v_{\alpha;\beta} - v_{\beta;\alpha}) - \frac{1}{2}(a_\alpha v_\beta - a_\beta v_\alpha).$$

The scalar measure of rotation

$$\omega^2 = \frac{1}{2}\omega^{\alpha\beta}\omega_{\alpha\beta}$$

Inhomogeneity

If one drops the idea of a spatially homogeneous universe, there could be "non-gravitatioal forces!"

This gives rise to an acceleration.

Example: If there is a spatial gradient of pressure, the motion is non-geodesic.

$\rightarrow$ A nonzero acceleration a^{μ}.

For a perfect fluid, $a^{\mu} = -\frac{p_{,\nu}(g^{\mu\nu} - v^{\mu}v^{\nu})}{\rho + p}$

Raychaudhuri equation

One can actually start from the commutator of covariant derivative of the velocity vector,

$$v^{\alpha}_{;\mu;\nu} - v^{\alpha}_{;\nu;\mu} = R^{\alpha}_{\beta\nu\mu} v^{\beta}$$

($R^{\alpha}_{\beta\nu\mu}$ is the Riemann tensor)

and arrive at the equation

$$\theta_{,\mu} v^{\mu} - a^{\mu}_{;\mu} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) = R_{\mu\nu} v^{\mu} v^{\nu}$$

which is the famous Raychaudhuri equation.

Important point to note: This equation utilizes equations of Riemannian geometry. General relativity is not yet invoked!!

Raychaudhuri equation

Where does General Relativity come into picture?

Replace The Ricci tensor from Einstein equation

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -8\pi G T_{\alpha\beta}$$

For a perfect fluid distribution, the velocity vector is an eigenvector of the Ricci tensor.
The right hand side of Raychaudhuri equation becomes:

$$-4\pi G(\rho + p)$$

Raychaudhuri equation

With a fluid like this, Raychaudhuri equation takes the form

$$\dot{\theta} + \frac{1}{3}\theta^2 = -4\pi G(\rho + p) + a^{\mu}_{;\mu} - 2(\sigma^2 - \omega^2)$$

where a dot represents a differentitation wrt the proper time.

$a \rightarrow$ measure of the linear dimension (average scale factor), the left hand side reads as

$$3\frac{\ddot{a}}{a}.$$

In order to avoid the singularity of zero dimension, a "minimum" is required.

$\ddot{a}$ should be positive at the extremum.

With valid energy conditions, there are only two possibilities:

- ω^2
- $a^{\mu}_{;\mu}$

The original version

The original version of Raychaudhuri equation[1] considers geodesic motion (no acceleration) in a dust universe ($p = 0$)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho - \sigma^2 + \frac{2}{3}\omega^2 + \frac{\Lambda}{3}.$$

Note: It includes Λ !

Raychaudhuri worked with the time-time component of Einstein equation (G^0_0) and also assumed a dust distribution for the sake of simplicity.

[1] A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955)

The original version

The general form that we see now, was given by Ehlers[2]. The main difference is

The inclusion of acceleration

More general form of matter distribution

Raychaudhuri was interested in geodesic motion: acceleration did not appear in his original equation.

[2] J. Ehlers, Akad. Wiss. Lit. Abhandl. Math.-Nat. Kl., **11**. 793 (1961); J. Ehlers, Ge. Relativ. Gravit. **25**, 1225 (1993)

Example with vorticity

Godel presented an exact solution of Einstein's equations in 1949:

$$ds^2 = a^2\left[(dt + e^x dy)^2 - dx^2 - \frac{e^{2x}}{2}dy^2 - dz^2\right]$$

This is stationary, has a constant density, and

does not have a singularity!

It has a non-zero vorticity, $\omega^2 = 4\pi G\rho = \frac{a^{-2}}{2}$

Also has a cosmological constant $\Lambda = -\frac{a^{-2}}{2}$

The solution is physically not acceptable!!

Has "closed time like lines", past and present are not properly distinguished.

Example with inhomogeneity

Spatial gradient of pressure $\rightarrow$ The motion is non-geodesic

A chance of avoiding the singularity through the divergence of acceleration.

Senovilla[3] (1990):

$$ds^2 = C^4(at)C^2(3ar)(dt^2 - dr^2) - \frac{1}{9a^2}C^4(at)S^2(3ar)C^{-\frac{2}{3}}(3ar)d\phi^2 - C^{-2}(at)C^{-\frac{2}{3}}(3ar)dz^2.$$

Energy condition $\rightarrow$ satisfied
Vorticity $\rightarrow$ None !

Acceleration $\rightarrow$ YES !

$$a_1 = 3aS(3ar)C^{-2}(at)C^{-2}(3ar)$$

Singularity $\rightarrow$ None!

[3] J.M.M. Senovilla, Phys. Rev. Lett. **64**, 2219 (1990)

The two examples: A quick revisit

The first example: Godel Solution

Godel solution escapes the singularity with the help of vorticity !

Defect: Closed time like lines! $\rightarrow$ Unphysical.

The two examples: A quick revisit

The second example: Senovilla's solution escapes singularity through acceleration (non-geodesic motion)

The spacetime is not isotropic (it has a cylindrical symmetry!)

The observed universe is isotropic; but no theoretical problem in this case.

Defect: The average of density, pressure and all physically relevant scalars $\rightarrow$ ZERO !!

This was also shown by AKR [4] in 1998 !!

[4] A.K. Raychaudhuri, Phys. Rev. Lett. **80**, 654 (1998).

Light rays?

Talked about timelike lines – motion of particles with mass.

How about massless particles like photons?

The corresponding equation was given by Sachs[5] in 1961.

The left hand side looks the same

The right hand side : $R_{\mu\nu} n^{\mu} n^{\nu}$,

$n^{\mu} \rightarrow$ a null vector.

One word of caution: the acceleration and the expansion scalar are composed of the derivatives of n^{μ}, and are a bit different.

[5] R.K. Sachs, Proc. R. Soc. (London), **A264**, 309 (1961); ibid **A270**, 103 (1962)

The Singularity: A more sophisticated look

Singularity $\rightarrow$ zero length scale $\rightarrow$ zero volume $\rightarrow$ infinite density, infinite pressure etc.

Actual interest: particle trajectories.

If a timelike line, or a null line is terminated at a point, the particle motion is restricted, and we call it a singularity in a more general sense.

If all lines are terminated at the same point we have a "focusing" of such lines.

In the absence of acceleration, the relevant lines of motion will be the geodesics.

The Singularity Theorem

The celebrated Hawking Penrose Singularity Theorem tells us:

A geodesic incompleteness is unavoidable, if

1) The energy condition is satisfied

$$(\rho + p) > 0$$

2) There are no closed time like lines

The proof of the theorem actually starts from Raychaudhuri equation.

(For a detail description see the monograph by Hawking and Ellis[6])

[6] S.W. Hawking and G.F.R. Ellis *The Large Scale Structure of Space Time*, Cambridge: Cambridge University Press (1973

A modern Assessment of the importance

It is the FUNDAMENTAL equation of gravitational attraction
- George Ellis[7]

The paper* reviewed here is truly a landmark in cosmology. It showed how results are in principle observable and local variables can be obtained without imposition of isometries. It can be considered a forerunner of a programme begun by J Kristian and R K Sachs and continued by G Ellis and his coworkers. Moreover, it paved the way to the PenroseHawking singularity theorems.
- Jurgen Ehlers[8]

* Relativistic Comology 1 - AKR, Phys. Rev 1955

[7] G.F.R. Ellis, Pramana - J. Phys., **69**, 15 (2007)
[8] J. Ehlers, Pramana - J. Phys., **69**, 7 (2007)

A modern assessment of importance

Further evidence was introduced in 1955 that, in the cosmological context at least, singularities in general relativistic spacetimes are not artifacts of symmetry assumption. The evidence came not from one of the havens of general relativity in America or Europe, but from Calcutta. In 1953 Raychaudhuri produced an analysis of cosmological singularities that was to have a profound effect on later developments but because it ran into difficulties with referees, the paper did not appear until 1955.

- John Earman[9]

[9] J. Earman, *The Expanding World of General Relativity*, Einstein Studies **7**, eds. H. Goenner *et al*, Barkhause Verlag, Boston (1998)

The battle alone

AKR had been working all alone in those days!!

He was working as a Lecturer at Ashutosh College, Calcutta

In 1953, he joined Indian Association for the Cultivation of Science, Calcutta as a Research Officer.

His expectation: He would have enough time for his work!

Reality:

- He was asked by the Director to work on Solid State Physics!
- Then he was asked to work on Quantum Chemistry!!

The battle alone

All are nice, but completely orthogonal to his likings!

He did both!

He extended Coulson's work on Free electron network approximation for pure metals. Coulson's idea was that it would not work for alloys. AKR showed that it would work for alloys as well! Coulson himself appreciated the work!

Not only the compulsion of doing something not of his interest, at some point of time he was asked to sit in a small room which was used as a kitchen!

But his interest in relativistic cosmology never abated!

A bit about the publication of the paper

The first version of the paper, which included some speculations as well, was sent to Physical Review as letter to the editor in April 1953.

The title of the paper was

"Local Temporal Behaviour of a Gravitating System in General Relativity".

It was promptly rejected. The summary of the referee's comment was that he could not understand where does the equation come from!

A bit about the publication of the paper

AKR then removed the speculative part, renamed it as “Relativistic Cosmology 1” and sent it to Physical Review as a regular paper in December 1953. There was no response from the journal, in spite of repeated queries from the author!

After about a year, the paper was sent to Zeitschrift fur Physik.

Prompt came the rejection from the editor, saying that there the results hardly had any impact!

In February, 1955, the editor of Physical review wrote that he could finally recover the report from the referee who recommended publication!

Resources!

The history of the publication of the paper can be found in the exchange of letters between John Earman and AKR in 1995.

The history of the days of the relentless fight can be found from AKR's diary.

Both these are included in an anthology of various writings of AKR *Atmajijnasa*, edited by Parongama Sen.

An exhaustive review of Raychaudhuri equation was written by Sayan Kar and Soumitra Sengupta, in the special issue of Pramana, **69**, 2007.